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TORSION OF A CIRCULAR CONE WITH STATIC AND DYNAMIC LOADING†

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Integral transformation methods—the Mellin transform for statics and the Lebedev–Kontorovich transform for dynamics—are used to construct analytic solutions of the problem of the torsion of an elastic circular cone. Assuming that external forces are concentrated in the neighbourhood of the vertex of the cone, the asymptotic behaviour of the far field is investigated. It is shown that the leading term of the asymptotic expansion is governed by the magnitude of the moment of the external forces, so that the St Venant principle is satisfied in the cases under consideration.

1. The static version of the problem of the torsion of a circular cone has been considered by many authors. References can be found in [1], which mathematically formulates the torsion problem for a circular cone with aperture angle 2α in a cylindrical system of coordinates (r, φ, z) with origin at the vertex of the cone. It shows that one can seek a solution of the problem using the displacement function $\psi(r, z) = r^{-1}\upsilon(r, z)$ which satisfies the equation

$$\frac{\partial^2\psi}{\partial r^2}+\frac{3}{r}\frac{\partial\psi}{\partial r}+\frac{\partial^2\psi}{\partial z^2}=0 \tag{1.1}$$

The shear force on the side surface of the cone is given in the form

$$T_\nu=\sigma_{r\varphi}\cos(r,\nu)+\sigma_{z\varphi}\cos(z,\nu)=\mu r\frac{\partial\psi}{\partial\nu} \tag{1.2}$$

$$\cos(r,\nu)=dr/d\nu=dz/ds=\cos(z,s)$$

$$\cos(z,\nu)=dz/d\nu=-dr/ds=-\cos(r,s)$$

(ν is the normal and s is the tangent unit vector to the contour of an axial section of the cone.) In [1] the solution of problem (1.1), (1.2), was also given, but the formal equation does not provide an answer to the question of whether this problem satisfies the St Venant principle. Below we establish the validity of the St Venant principle in the problem of torsion of a circular cone for static and dynamic loads. At the same time, expressions are obtained for the leading terms of the asymptotic expansion of the solution for large r.

To solve Eq. (1.1) we change to the spherical system of coordinates (ρ, φ, θ), for which $r = \rho\sin\theta$, $z = \rho\cos\theta$. Equation (1.1) acquires the form

$$\frac{\partial^2\psi}{\partial\rho^2}+\frac{4}{\rho}\frac{\partial\psi}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\theta^2}+\frac{3\operatorname{ctg}\theta}{\rho^2}\frac{\partial\psi}{\partial\theta}=0 \tag{1.3}$$

and the boundary condition, using (2.1) reduces to the following

$$\theta=\alpha,\quad \sigma_{\theta\varphi}=2\mu e_{\theta\varphi}=\frac{\mu}{\rho}\left(\frac{\partial\upsilon}{\partial\theta}-\upsilon\operatorname{ctg}\theta\right)=\mu\sin\theta\frac{\partial\psi}{\partial\theta}=f(\rho) \tag{1.4}$$

The solution of boundary-value problem (1.3), (1.4) can be represented in the form of a Mellin contour integral

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$$\psi(\rho,\theta) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\bar{\psi}(s,\theta)\rho^{-s}ds \tag{1.5}$$

Here, because of energy conditions, the constant c must lie in the interval (0, 1). Then for the function $\bar{\psi}(s,\theta)$ we have the equation and boundary condition

$$\frac{d^2\bar{\psi}}{d\theta^2} + 3\operatorname{ctg}\theta\frac{d\bar{\psi}}{d\theta} + s(s-3)\bar{\psi} = 0 \tag{1.6}$$

$$\theta = \alpha, \quad \mu\sin\alpha\frac{d\bar{\psi}}{d\theta} = \bar{f}(s) = \int_0^\infty f(\rho)\rho^{s-1}d\rho \tag{1.7}$$

The substitution $\bar{\psi}(s,\theta) = y'_\xi(n,\xi)$, $\xi = \cos\theta$, $s(s-3) = n^2 + n - 2$ enables us to reduce (1.6) to the Legendre equation

$$(\xi^2-1)y''_{\xi\xi} + 2\xi y'_\xi - n(n+1)y = 0$$

Hence, for a continuous circular cone, using boundary condition (1.7) and well-known notation [2] we have

$$\bar{\psi}(s,\theta) = \frac{-\bar{f}(s)}{\mu\sin^2\alpha}\frac{P'_n(\xi)\big|_{\xi=\cos\theta}}{P'_n(\xi)\big|_{\xi=\cos\alpha}} = \frac{\bar{f}(s)P_n^1(\cos\theta)}{i\mu\sin\theta P_n^2(\cos\alpha)}$$

where $n = s-2$. We note the existence of two possible relations between n and s, but by the properties of spherical functions we have $P_{s-2}^k = P_{1-s}^k$, and the second version of the relation between n and s need not be considered below.

We obtain the representation

$$\psi(\rho,\theta) = -\frac{1}{2\pi\mu\sin\theta}\int_{c-i\infty}^{c+i\infty}\frac{\bar{f}(s)P_{s-2}^1(\cos\theta)}{P_{s-2}^2(\cos\alpha)}\rho^{-s}ds \tag{1.8}$$

for the displacement function. We obtain the asymptotic behaviour of solution (1.8) when $\rho \gg 1$ by completing the contour of integration $(c - i\infty, c + i\infty)$ with a semicircle of infinite radius in the half-plane Re $s > 0$ and calculating the integral using the residue theorem. In the first approximation we obtain

$$\psi(\rho,\theta) = \frac{B(\theta)\bar{f}(3)}{\rho^3} + \sum_{k=1}^{\infty}\frac{A_k(\theta)\bar{f}(s_k)}{\rho^{s_k}} \tag{1.9}$$

where the s_k are the roots of the equation

$$P_{s-2}^2(\cos\alpha) = 0 \tag{1.10}$$

One root of Eq. (1.10) is obvious $s = 3$. Indeed, because $P_1^1(\xi)$ is a first-degree polynomial so that $P_1'(\xi) = 0$, then when $|\xi| < 1$ we have $P_1^2(\xi) = 0$.

Theorem. For any $\theta \in (0, \pi)$ the equation $P_n^2(\cos\theta) = 0$ has no roots in the domain $0 \leqslant \text{Re} \leqslant 2$ that are different from $n = 0, 1, 2$.

Proof. 1. For fixed z the function $P_n^2(s)$ only has real zeros. Indeed, for integer m we have the formula [2]

$$P_n^{-m}(z) = \frac{\Gamma(n-m+1)}{\Gamma(n+m+1)}P_n^m(z) \tag{1.11}$$

By virtue of (1.11), if $P_n^2(z)$ has complex zeros, then they are also zeros of $P_n^{-2}(z)$. But we know [2] that $P_n^{-m}(z)$ has no complex zeros in n when $m \geqslant 0$.

2. We use the formula [2]

$$P_n^2(\cos\theta) = -\frac{4i\operatorname{sh}\theta}{\Pi(-\frac{1}{2})\Pi(-\frac{5}{2})}\int_0^\infty \frac{2\sin\pi n\operatorname{ch}(n+\frac{1}{2})tdt}{(2\operatorname{ch}t+2\cos\theta)^{5/2}} \tag{1.12}$$

One can verify that when $|\operatorname{Re} n| < 2$ the integral (1.12) converges. Then, if $P_n^2(\cos\theta) = 0$ for any $n \neq 1$ lying in the above interval, then

$$\int_0^\infty \frac{\operatorname{ch}(n+\frac{1}{2})tdt}{(\operatorname{ch}t+2\cos\theta)^{5/2}} = 0 \tag{1.13}$$

and this is impossible for real n because the integrand is positive.

The theorem implies that the leading term of the asymptotic expansion for $\rho \gg 1$ in formula (1.9) corresponds to $n = 1$ or, equivalently, $s = 3$, and is equal to $B(\theta)\bar{f}(s)\rho^{-3}$, i.e. apart from a multiplicative constant, it gives the moment of the tangential stresses applied to the side surface of the cone.

We shall assume that the function $f(\rho)$ is non-zero in the neighbourhood of the vertex of the cone. Then our analysis confirms the validity of the St Venant principle in the present problem: far from the vertex the stress–strain state of the cone is mainly given by the resultant moment of the applied surface forces and does not depend on their distribution. Thus, the torsion problem for a circular cone is not a singular problem in the terminology previously proposed [3].

2. In the case when the side surface of the cone is dynamically loaded by forces that vary harmonically with time, we obtain the following boundary-value problem for the displacement $\upsilon(\rho, \theta)$

$$\frac{\partial^2\upsilon}{\partial\rho^2}+\frac{2}{\rho}\frac{\partial\upsilon}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2\upsilon}{\partial\theta^2}+\frac{1}{\rho^2}\operatorname{ctg}\frac{\partial\upsilon}{\partial\theta}+\left(k^2-\frac{1}{\rho^2\sin\theta}\right)\upsilon=0 \tag{2.1}$$

$$\theta=\alpha,\quad \frac{\mu}{\rho}\left(\frac{\partial\upsilon}{\partial\theta}-\upsilon\operatorname{ctg}\theta\right)=f(\rho) \tag{2.2}$$

Problems of this form [4] have been studied using the integral transformation

$$\bar{\upsilon}(s,\theta)=\int_0^\infty \rho^{1/2}\upsilon(\rho,\theta)H_s^{(2)}(k\rho)d\rho \tag{2.3}$$

The inverse transformation has the form

$$\upsilon(\rho,\theta)=-\frac{1}{2\sqrt{\rho}}\int_{-i\infty}^{i\infty} s\bar{\upsilon}(s,\theta)J_s(k\rho)/\,ds \tag{2.4}$$

From (2.1) there follows an equation for $\bar{\upsilon}(s, \theta)$ whose solution is expressed as s sum of Legendre functions

$$\bar{\upsilon}(s,\theta)=A(s)P^1_{s-1/2}(\cos\theta)+B(s)P^1_{s-1/2}(-\cos\theta) \tag{2.5}$$

It follows from the finiteness of the cone displacements that the constant $B(s)$ is zero. We compute the function $A(s)$ from the boundary condition (2.2) and obtain

$$A(s)=\frac{i}{\nu\Delta(s-\frac{1}{2})}\int_0^\infty f(r)\sqrt{r}H_s^{(2)}(kr)dr \tag{2.6}$$

$$\Delta(s)=i\left[\frac{d}{d\theta}P_s^1(\cos\theta)-\operatorname{ctg}\theta P_s^1(\cos\theta)\right]_{\theta=\alpha}=P_s^2(\cos\alpha). \tag{2.7}$$

where the last equality follows from a well-known result [2]. Substituting (2.6) into (2.4) and changing the order of integration, we obtain the integral representation

$$\upsilon(\rho,\theta)=\frac{1}{2\mu}\int_0^\infty K(k\rho,kr,\theta)f(r)dr \tag{2.8}$$

$$K(k\rho,kr,\theta)=-i\sqrt{\frac{r}{\rho}}\int_{-i\infty}^{i\infty}\frac{sP^1_{s-1/2}(\cos\theta)H^{(2)}_s(k\rho)J_s(kr)ds}{P^2_{s-1/2}(\cos\alpha)} \tag{2.9}$$

for the displacement $\upsilon(\rho, \theta)$ and an expression similar to (2.9) which is obtained by the circular replacement $r \to \rho \to r$ in the integrand.

Integral (2.9) is calculated using the residue theorem

$$K(k\rho,kr,\theta)=2\pi\sqrt{\frac{r}{\rho}}\sum_{\operatorname{Re}s_\nu>-1/2}(s_\nu+1/2)\frac{P^1_{s_\nu}(\cos\theta)H^{(2)}_{s_\nu+1/2}(k\rho_1)J_{s_\nu+1/2}(k\rho_2)}{\Delta'(s_\nu)}$$

where

$$\rho_1=\max(\rho,r),\quad \rho_2=\min(\rho,r)$$

s_ν are roots of the equation $P^2_{s_\nu}(\cos\alpha) = 0$, which, as was established above, has no other roots in the domain $-1/2 < \operatorname{Re} s_\nu < 3/2$ other than $s = s_1 = 1$.

We assume that $f(r) = 0$ when $r > \delta > 0$, with $k\delta \ll 1$. Then for $\rho \gg \delta$ we have the asymptotic relation

$$K(k\rho,kr,\theta)=\frac{3\pi P^1_1(\cos\theta)}{\Gamma(5/2)\Delta'(1)}\left(\frac{k}{2}\right)^{3/2}\frac{r^2H^{(2)}_{3/2}(k\rho)}{\sqrt{\rho}}+o(\delta^2) \tag{2.10}$$

It follows from (2.8) and (2.10) that if the torsional oscillations of the cone are excited by forces concentrated in the neighbourhood of the vertex of the cone that is small compared to the wavelength, then the leading term of the generated displacement field is described by the expression

$$MC(k)(k\rho)^{-1/2}H^{(2)}_{3/2}(k\rho)P^1_1(\cos\theta),\quad C(k)=\frac{3}{\sqrt{2}\Gamma(5/2)\Delta'(1)}\left(\frac{k}{2}\right)^2$$

depending only on the total torsional moment

$$M=\int_0^\delta f(r)r^2dr$$

The St Venant principle is therefore also satisfied in the dynamic problem of the torsion of a circular elastic cone.

REFERENCES

1. ARUTYUNYAN N. Kh. and ABRAMYAN B. L., *Torsion of Elastic Bodies*. Fizmatgiz, Moscow, 1963.
2. HOBSON E. V., *The Theory of Spherical and Ellipsoidal Functions*. IIL, Moscow, 1952.
3. SANCHEZ-PALENCIA E., Some problems of the theory of elasticity in unbounded domains and their application to contact problems. *Tr. Mat. Inst. Akad. Nauk SSSR* **192,** 183–196, 1990.
4. LEBEDEV N. N., and KONTOROVICH M. I., On the application of inversion formulae to some problems in electrodynamics. *Zh. Eksp. Tevr. Fiz.* **9,** 6, 729–741, 1939.
5. SMIRNOV V. I., *Advanced Mathematics*, Vol. 3, Pt 2. Nauka, Moscow, 1974.

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